HIGGS BUNDLES FOR THE LORENTZ GROUP

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Abstract. Using the Morse-theoretic methods introduced by Hitchin, we prove that the moduli space of $\mathrm{SO}_0(1,n)$-Higgs bundles when n is odd has two connected components.

Introduction

Let G be a real semisimple Lie group and let $H \subseteq G$ be a maximal compact subgroup. Let $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$ be the complexified isotropy representation defined in terms of the Cartan decomposition of the Lie algebra of G. Let X be a compact Riemann surface of genus $g \geq 1$. A *G-Higgs bundle* over X is a pair (E, φ) consisting of a principal $H^{\mathbb{C}}$-bundle E over X and a holomorphic section φ of the bundle associated to ι twisted by the canonical line bundle of X. For these objects there is a notion of (poly)stability that allows to construct the moduli space of isomorphism classes of polystable G-Higgs bundles. Higgs bundles were introduced by Hitchin in [12, 13] when G is complex and in [14] when G is the split real form of a complex semisimple Lie group. Other real forms, especially of Hermitian type have been studied in [2, 4, 8] and other papers.

In [1] a systematic study has been initiated for $G = \mathrm{SO}_0(p,q)$ — the connected component of the identity of $\mathrm{SO}(p,q)$. In this paper we report on the solution to the problem of counting the number of connected components of the moduli space of polystable $\mathrm{SO}_0(1,n)$-Higgs bundles when n is odd. We prove the following.

Theorem (see Theorem 9.3). *The moduli space of* $\mathrm{SO}_0(1,n)$*-Higgs bundles when* $n > 1$ *is odd has two connected components.*

An important motivation to study G-Higgs bundles comes from their relation with representations of the fundamental group of the surface X in G. Namely, for a semisimple algebraic Lie group G we say that a representation of $\pi_1(X)$ in G — that is a homomorphism of $\pi_1(X)$ in G — is reductive if the Zariski closure of its image is a reductive group. The moduli space of equivalence classes of reductive representations is an algebraic variety [10]. Non-abelian Hodge theory [5, 6, 7, 8, 12, 18, 19] says precisely that this variety is homeomorphic to the moduli space of polystable G-Higgs bundles. We thus have the following as a corollary of our main theorem.

Theorem. *The moduli space of reductive representations of the fundamental group of an orientable compact surface in* $\mathrm{SO}_0(1,n)$ *when* $n > 1$ *is odd has two connected components.*

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The main tool to prove our result is the use of the Morse-theoretic techniques introduced by Hitchin [12, 14]. These techniques have by now been used to count the number of connected components of the moduli space of G-Higgs bundles for several groups (see e.g. [2, 3, 11, 9, 8, 15]). A main step is to identify the critical subvarieties of the Hitchin-Morse function defined by the L^2-norm of the Higgs field. This has been carried out in [1] in full generality for $\mathrm{SO}_0(p,q)$. Now, the problem of identifying the local minima — which is what allows the counting of connected components — in general is far more involved technically than for the other groups studied in the literature. This is however possible for $\mathrm{SO}_0(1,n)$ when n is odd. The main technical bulk of the paper is devoted to identifying in this case, first the smooth minima in the moduli space, and then the possibly singular points, which consist of stable but not simple Higgs bundles and strictly polystable Higgs bundles. We expect that our results may be of interest both in geometry and physics since $\mathrm{SO}(1,n)$ is the Lorentz group of special relativity and its adjoint form is the group of isometries of real hyperbolic space.
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1. $\mathrm{SO}_0(1,n)$-Higgs bundles

Let X be a compact Riemann surface. Let G be a real semisimple Lie group, H be a maximal compact subgroup of G and $H^{\mathbb{C}}$ be its complexification. Let

$$\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}}),$$

be the complexified isotropy representation, defined in terms of the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra of G and using the fact that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Definition 1.1. A **G-Higgs bundle** is a pair (E, φ) where E is a principal $H^{\mathbb{C}}$-bundle over X and φ is a holomorphic section of the vector bundle $E(\mathfrak{m}^{\mathbb{C}}) \otimes K = (E \times_\iota \mathfrak{m}^{\mathbb{C}}) \otimes K$, where K is the canonical line bundle over X. The section φ is called the **Higgs field**.

When G is a real compact reductive Lie group, the Cartan decomposition of the Lie algebra is $\mathfrak{g} = \mathfrak{h}$ and then the Higgs field is equal to zero. Hence, a G-Higgs bundle is in fact a principal $G^{\mathbb{C}}$-bundle.

If G is a complex Lie group, we consider the underlying real Lie group $G^{\mathbb{R}}$. In this case, the complexification $H^{\mathbb{C}}$ of a maximal compact subgroup is again the Lie group G and since

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{h} + i\mathfrak{h},$$

the isotropy representation coincides with the adjoint representation of G on its Lie algebra.

The special orthogonal group $\mathrm{SO}(1,n)$ is the subgroup of $\mathrm{SL}(n+1,\mathbb{R})$ consisting of all linear transformations of a $n+1$ dimensional real vector space which leave invariant a non-degenerate symmetric bilinear form of signature $(1,n)$. Using the standard non-degenerate symmetric bilinear form of signature $(1,n)$ on $\mathbb{R}^{n+1}$

$$\epsilon(x,y) = -x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1},$$

this means that,

$$\mathrm{SO}(1,n) = \{A \in \mathrm{SL}(n+1,\mathbb{R}) \mid A^t I_{1,n} A = I_{1,n}\},$$

where $I_{1,n} = \begin{pmatrix} -1 & \\ & I_n \end{pmatrix}$.

The Lie group $\mathrm{SO}(1,n)$ is a non-compact real form of $\mathrm{SO}(n+1,\mathbb{C})$. It has dimension $n(n+1)/2$, is semisimple for $n \geq 2$ and has two connected components. Let $\mathrm{SO}_0(1,n)$ be the connected component of the identity.

The Lie algebra of $\mathrm{SO}(1,n)$ and then of its identity component $\mathrm{SO}_0(1,n)$ is $\mathfrak{so}(1,n)$, which has Cartan decomposition

$$\mathfrak{so}(1,n) = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{h} = \mathfrak{so}(n)$ is the Lie algebra of the maximal compact subgroup $\mathrm{SO}(1) \times \mathrm{SO}(n)$ of $\mathrm{SO}_0(1,n)$. If we use the standard non-degenerate symmetric bilinear form of signature $(1,n)$, we have that

$$\begin{aligned} \mathfrak{so}(1,n) &= \{X \in \mathfrak{sl}(n+1,\mathbb{R}) \mid X^t I_{1,n} + I_{1,n} X = 0\} \\ &= \left\{ \begin{pmatrix} 0 & X_2 \\ X_2^t & X_3 \end{pmatrix} \mid X_3 \text{ real skew-sym. of rank } n,\ X_2 \in \mathbb{R}^n \right\}, \end{aligned}$$

and then

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & X_3 \end{pmatrix} \mid X_3 \in \mathfrak{so}(n) \right\},$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & X_2 \\ X_2^t & 0 \end{pmatrix} \mid X_2 \in \mathbb{R}^n \right\}.$$

The involution of $\mathfrak{so}(n+1,\mathbb{C})$ that defines $\mathfrak{so}(1,n)$ as a real form is $\sigma(X) = I_{1,n}\bar{X}I_{1,n}$, that is

$$\begin{aligned} \mathfrak{so}(1,n) &= \{X \in \mathfrak{so}(n+1,\mathbb{C}) \mid I_{1,n}\bar{X}I_{1,n} = X\} \\ &= \{X \in \mathfrak{sl}(n+1,\mathbb{C}) \mid X + X^t = 0, I_{1,n}\bar{X}I_{1,n} = X\} \\ &= \left\{ \begin{pmatrix} 0 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix} \mid X_3 \text{ real skew-sym. of rank } n,\ X_2 \in \mathbb{R}^n \right\}. \end{aligned}$$

Observe that there is an isomorphism

$$\begin{pmatrix} 0 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix} \to \begin{pmatrix} 0 & X_2 \\ X_2^t & X_3 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & I_n \end{pmatrix}.$$

The Cartan decomposition of the complex Lie algebra is

$$\mathfrak{so}(n+1,\mathbb{C}) = \mathfrak{so}(n,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}},$$

where

$$\mathfrak{m}^{\mathbb{C}} = \{ \begin{pmatrix} 0 & X_2 \\ -X_2^t & 0 \end{pmatrix} \mid X_2 \in \mathbb{C}^n \},$$

and the complexified isotropy representation is

$$\iota : \{1\} \times \mathrm{SO}(n,\mathbb{C}) \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}}),$$

where

$$\iota\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & X_2 \\ -X_2^t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & X_2 \\ -X_2^t & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & X_2 b^{-1} \\ -bX_2^t & 0 \end{pmatrix} \in \mathfrak{m}^{\mathbb{C}}.$$

From Definition 1.1, an $\mathrm{SO}_0(1,n)$**-Higgs bundle** is a pair (E,φ) consisting of a holomorphic principal $\mathrm{SO}(1,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$-bundle E over X and a holomorphic section $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}})\otimes K)$.

If (E,φ) is an $\mathrm{SO}_0(1,n)$-Higgs bundle, the principal $\mathrm{SO}(1,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$-bundle E is the fibred product

$$E = E_{\mathrm{SO}(1,\mathbb{C})} \times E_{\mathrm{SO}(n,\mathbb{C})}$$

of two principal bundles with structure groups $\mathrm{SO}(1,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$ respectively. Using the standard representations of $\mathrm{SO}(1,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$ in $\mathbb{C}$ and $\mathbb{C}^n$ we can associate to E a triple (V,W,Q_W) where $V\cong\mathcal{O}$, W is a holomorphic vector bundle of rank n and trivial determinant, $Q_W : W\otimes W\to\mathbb{C}$ is a non-degenerate symmetric quadratic form, which induces an isomorphism $q_W : W \xrightarrow{\sim} W^*$.

The vector bundle $E(\mathfrak{m}^{\mathbb{C}})$ can be expressed in terms of $V\cong\mathcal{O}$ and W as follows:

$$E(\mathfrak{m}^{\mathbb{C}}) = \{(\eta,\nu)\in \mathrm{Hom}(W,\mathcal{O})\oplus\mathrm{Hom}(\mathcal{O},W) \mid \nu = -\eta^\top\},$$

where $\eta^\top = q_W^{-1}\circ\eta^t$,

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\eta^\top"] \arrow[dr, "\eta^t"'] & W \arrow[d, "q_W"] \\ & W^*, \end{tikzcd}$$

that is, $E(\mathfrak{m}^{\mathbb{C}})\cong\mathrm{Hom}(W,\mathcal{O})$. Then, in terms of vector bundles, the Higgs field is a section $\eta\in H^0(\mathrm{Hom}(W,\mathcal{O})\otimes K)$, that is

$$\eta : W \to \mathcal{O}\otimes K,$$

and hence $\mathrm{SO}_0(1,n)$-Higgs bundles (E,φ) are in one-to-one correspondence with tuples $(\mathcal{O},W,Q_W,\eta)$.

Let (E,φ) be an $\mathrm{SO}_0(1,n)$-Higgs bundle. Extending the structure group of E from $\mathrm{SO}(1,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})$ to $\mathrm{SO}(n+1,\mathbb{C})$, the pair $(E_{\mathrm{SO}(n+1,\mathbb{C})},\varphi)$, with

$$\varphi\in H^0(E_{\mathrm{SO}(1,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C})}(\mathfrak{m}^{\mathbb{C}})\otimes K)\subset H^0(E_{\mathrm{SO}(n+1,\mathbb{C})}(\mathfrak{so}(n+1,\mathbb{C}))\otimes K),$$

is an $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle.

In terms of vector bundles, if $\mathbf{E}$ is the vector bundle associated to $E_{\mathrm{SO}(n+1,\mathbb{C})}$ via the standard representation of $\mathrm{SO}(n+1,\mathbb{C})$ in $\mathbb{C}^{n+1}$ and $(\mathcal{O},W,Q_W,\eta)$ is the tuple corresponding to (E,φ), then $\mathbf{E}=\mathcal{O}\oplus W$, and the $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle associated to $(\mathcal{O},W,Q_W,\eta)$ is the triple

$$(\mathbf{E}=\mathcal{O}\oplus W, Q=\begin{pmatrix} 1 & \\ & Q_W \end{pmatrix}, \phi=\begin{pmatrix} & \eta \\ -\eta^\top & \end{pmatrix}).$$

2. Stability conditions

In this section we study the notions of semistability, stability and polystability for $\mathrm{SO}_0(1,n)$-Higgs bundles, for the associated $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundles and the relation between them. These notions have been studied in [1] applying the general notions given by Bradlow, García-Prada, Gothen and Mundet i Riera [5, 8], that generalize the results given by Ramanathan [17] for principal bundles.

We will use these notions in term of filtrations. In the case of $\mathrm{SO}_0(1,n)$-Higgs bundles, since $\mathrm{SO}(1,\mathbb{C}) = \{1\}$, they will only involve conditions on the filtrations of the principal $\mathrm{SO}(n,\mathbb{C})$-Higgs bundle (W, Q_W).

Definition 2.1. Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle with $n \neq 2$, then it is **semistable** if for any filtration

$$\mathcal{W} = (0 \subset W_1 \subset \cdots \subset W_s = W),$$

satisfying $W_j = V_{s-j}^{\perp_{Q_W}}$ and any element $\mu \in \Lambda(\mathcal{W})$ with

$$\Lambda(\mathcal{W}) = \{\mu = (\mu_1, \mu_2, \ldots, \mu_s) \in \mathbb{R}^s \mid \mu_i \leq \mu_{i+1}, \mu_{s-i+1} + \mu_i = 0 \text{ for any } i\},$$

such that $\eta \in H^0(N \otimes K)$, where

$$N = N(\mathcal{W}, \mu) = \sum_{\mu_i \geq 0} \mathrm{Hom}(W_i, \mathcal{O}),$$

we have

$$d(\mathcal{W}, \mu) \geq 0.$$

The tuple $(\mathcal{O}, W, Q_W, \eta)$ is **stable** if it is semistable and for any choice of the filtration $\mathcal{W}$ and non-zero $\mu \in \Lambda(\mathcal{W})$, such that $\eta \in H^0(N \otimes K)$, we have

$$d(\mathcal{W}, \mu) > 0.$$

Finally, the tuple $(\mathcal{O}, W, Q_W, \eta)$ is **polystable** if it is semistable and for any filtration $\mathcal{W}$ as above and non-zero $\mu \in \Lambda(\mathcal{W})$ satisfying $\mu_i < \mu_{i+1}$ for each i, $\eta \in H^0(N \otimes K)$ and $d(\mathcal{W}, \mu) = 0$, there is a splitting

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W/W_{s-1}$$

satisfying

$$Q_W(W_i/W_{i-1}, W_j/W_{j-1}) = 0 \text{ unless } i + j = s + 1,$$

with respect to which

$$\eta \in H^0(\bigoplus_{\mu_i = 0} \mathrm{Hom}(W_i/W_{i-1}, \mathcal{O}) \otimes K).$$

Definition 2.2. The **moduli space of polystable** $\mathrm{SO}_0(1,n)$**-Higgs bundles** is defined as the set of isomorphisms classes of polystable $\mathrm{SO}_0(1,n)$-Higgs bundles and is denoted by $\mathcal{M}(\mathrm{SO}_0(1,n))$.

In the following proposition we prove that the notions of semistability and stability can be simplified.

Proposition 2.3. *Let* $(\mathcal{O}, W, Q_W, \eta)$ *be an* $\mathrm{SO}_0(1,n)$*-Higgs bundle with* $n \neq 2$*. It is* ***semistable*** *if and only if for any isotropic subbundle* $W' \subset W$ *such that* $\eta(W') = 0$ *the inequality* $\deg W' \leq 0$ *holds. It is* ***stable*** *if and only if it is semistable and for any non-zero isotropic subbundle* $W' \subset W$ *such that* $\eta(W') = 0$ *we have* $\deg W' < 0$.

Proof. Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle and assume that for any isotropic subbundle $W' \subset W$ such that $\eta(W') = 0$, we have $\deg W' \leq 0$ holds. We want to prove that $(\mathcal{O}, W, Q_W, \eta)$ is semistable.

Choose a filtration $\mathcal{W} = (0 \subset W_1 \subset \ldots \subset W_s = W$ satisfying $W_j = W_{s-j}^{\perp_{Q_W}}$ for any j. We have to understand the geometry of the convex set

$$\Lambda = \{\mu \in \Lambda(\mathcal{W}) \mid \eta \in N\} \subset \mathbb{R}^s.$$

Let

$$\mathcal{J} = \{i \mid \eta(W_i) = 0\} = \{i_1, \ldots, i_k\}.$$

One checks easily that if $\mu \in \Lambda(\mathcal{W})$, then

$$\mu \in \Lambda \Leftrightarrow \mu_a = \mu_b, \text{ for any } i_l \leq a \leq b \leq i_{l+1}.$$

The set of indices $\mathcal{J}$ is symmetric, that is

$$i \in \mathcal{J} \Leftrightarrow s - i \in \mathcal{J}.$$

Let $\mathcal{J}' = \{i \in \mathcal{J} \mid 2i \leq s\}$ and define for any $i \in \mathcal{J}'$ the vector

$$L_i = -\sum_{c \leq i} e_c + \sum_{d \geq s-i+1} e_d,$$

where $\{e_1, \ldots, e_s\}$ are the canonical basis of $\mathbb{R}^s$. The set Λ is the positive span of the vectors $\{L_i \mid i \in \mathcal{J}'\}$ and we have that

$$d(\mathcal{W}, \mu) \geq 0 \text{ for any } \mu \in \Lambda \Leftrightarrow d(\mathcal{W}, L_i) \geq 0 \text{ for any } i.$$

We also have that

$$d(\mathcal{W}, L_i) = -\deg W_{s-i} - \deg W_i.$$

Since $\deg W_{s-i} = \deg W_i$, then $d(\mathcal{W}, L_i) = -2\deg W_i \geq 0$ is equivalent to $\deg W_i \leq 0$, which holds by assumption. Hence $(\mathcal{O}, W, Q_W, \eta)$ is semistable.

Conversely, if $(\mathcal{O}, W, Q_W, \eta)$ is semistable, for any isotropic subbundle $W' \subset W$ such that $\eta(W') = 0$ we have that the condition $\deg W' \leq 0$ is immediately satisfied by applying the semistability condition of the filtration $0 \subset W' \subset W'^{\perp_{Q_W}} \subset W$.

Finally, the proof of the second statement on stability is very similar to case of semistability and we then omit it. □

Remark 2.4. The case $n = 2$ requires special attention. Observe that a principal $\mathrm{SO}(2,\mathbb{C})$-bundle (E, Q) decomposes as $E = L \oplus L^{-1}$, where L is a line bundle and $Q = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Then, any principal $\mathrm{SO}(2,\mathbb{C})$-bundle has an isotropic subbundle with degree greater or equal than zero. However, $\mathrm{SO}(2,\mathbb{C}) \cong \mathbb{C}^*$ has no proper parabolic subgroups, and the stability condition can not be simplified in terms of isotropic subbundles. It seems that this case was overlooked in [17].

We now study the relation between the stability of an $\mathrm{SO}_0(1,n)$-Higgs bundle and the stability of its associated $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle. To do this we introduce the notions of semistability, stability and polystability for $\mathrm{SO}(n,\mathbb{C})$-Higgs bundles.

Definition 2.5. An $\mathrm{SO}(n,\mathbb{C})$-Higgs bundle $(\mathbf{E},Q,\phi)$ with $n\neq 2$ is **semistable** if for any filtration

$$\mathcal{E}=(0\subset E_1\subset\ldots\subset E_k=\mathbf{E}),$$

$1\leq k\leq n$, satisfying $E_j=E_{k-j}^{\perp_Q}$, and any element of

$$\Lambda(\mathcal{E})=\{\lambda=(\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_k)\in\mathbb{R}^k\mid\lambda_{k-i+1}+\lambda_i=0\text{ for any }i\}$$

such that $\phi\in H^0(N(\mathcal{E},\lambda)\otimes K)$, where

$$N(\mathcal{E},\lambda)=\mathfrak{so}(\mathbf{E})\cap\sum_{\lambda_j\leq\lambda_i}\mathrm{Hom}(E_i,E_j),$$

we have

$$d(\mathcal{E},\lambda)=\sum_{j=1}^{k-1}(\lambda_j-\lambda_{j+1})\deg E_j\geq 0.$$

The triple $(\mathbf{E},Q,\phi)$ is **stable** if it is semistable and for any choice of the filtration $\mathcal{E}$ and non-zero $\lambda\in\Lambda(\mathcal{E})$ such that $\phi\in H^0(N(\mathcal{E},\lambda)\otimes K)$, we have

$$d(\mathcal{E},\lambda)>0.$$

Finally, the triple $(\mathbf{E},Q,\phi)$ is **polystable** if it is semistable and for any filtration $\mathcal{E}$ as above and $\lambda\in\Lambda(\mathcal{E})$ satisfying $\lambda_i<\lambda_{i+1}$ for each i, $\phi\in H^0(N(\mathcal{E},\lambda)\otimes K)$ and $d(\mathcal{E},\lambda)=0$, there is an isomorphism

$$\mathbf{E}\simeq E_1\oplus E_2/E_1\oplus\cdots\oplus E_k/E_{k-1}$$

satisfying

$$Q(E_i/E_{i-1},E_j/E_{j-1})=0\text{ unless }i+j=k+1.$$

Furthermore, via this isomorphism,

$$\phi\in H^0(\bigoplus_i\mathrm{Hom}(E_i/E_{i-1},E_i/E_{i-1})\otimes K).$$

There is a simplification of the semistability and stability conditions, which is next described.

Proposition 2.6. *An $\mathrm{SO}(n,\mathbb{C})$-Higgs bundle $(\boldsymbol{E},Q,\phi)$ with $n\neq 2$ is **semistable** if and only if for any isotropic subbundle $E'\subset\boldsymbol{E}$ such that $\phi(E')\subseteq E'\otimes K$ the inequality $\deg E'\leq 0$ holds, and it is **stable** if it is semistable and for any non-zero isotropic subbundle $E'\subset\boldsymbol{E}$ such that $\phi(E')\subseteq E'\otimes K$ we have $\deg E'<0$.*

Proof. This proof is analogous to the proof of Theorem 3.9 in [8].

Let $(\mathbf{E},Q,\phi)$ be an $\mathrm{SO}(n,\mathbb{C})$-Higgs bundle and assume that for any isotropic subbundle $E'\subset\mathbf{E}$ such that $\phi(E')\subseteq E'\otimes K$ one has $\deg E'\leq 0$. We are going to prove that $(\mathbf{E},Q,\phi)$ is semistable.

Choose any filtration $\mathcal{E}=(0\subset E_1\subset\ldots\subset E_k=\mathbf{E})$ satisfying $E_j=E_{k-j}^{\perp_Q}$ for any j and consider the set

$$\Lambda(\mathcal{E},\phi)=\{\lambda\in\Lambda(\mathcal{E})\mid\phi\in N(\mathcal{E},\lambda)\}\subset\mathbb{R}^k.$$

Let $\mathcal{J} = \{j \mid \phi(E_j) \subseteq E_j \otimes K\} = \{j_1, \ldots, j_r\}$. One checks easily that if $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda(\mathcal{E})$ then

$$\lambda \in \Lambda(\mathcal{E}, \phi) \Leftrightarrow \lambda_a = \lambda_b \text{ for any } j_i \leq a \leq b \leq j_{i+1}.$$

The set of indices $\mathcal{J}$ is symmetric, i.e.,

$$j \in \mathcal{J} \Leftrightarrow k - j \in \mathcal{J}.$$

To check this we have to prove that $\phi(E_j) \subseteq E_j \otimes K$ implies that $\phi(E_j^{\perp_Q}) \subseteq E_j^{\perp_Q} \otimes K$. Suppose that this is not true, then there is a j with $\phi(E_j) \subseteq E_j \otimes K$ and there exists some $w \in E_j^{\perp_Q}$ such that $\phi(w) \notin E_j^{\perp_Q} \otimes K$. Then there exists $v \in E_j$ such that $Q(v, \phi(w)) \neq 0$. However, since $\phi \in H^0(\mathfrak{so}(\mathbf{E}) \otimes K)$, we must have

$$Q(v, \phi(w)) = Q(v, -\phi^\top(w)) = -Q(\phi(v), w),$$

and the latter vanishes because by assumption $\phi(v)$ belongs to E_j. So we have reached a contradiction.

Let $\mathcal{J}' = \{j \in \mathcal{J} \mid 2j \leq k\}$ and define for any $j \in \mathcal{J}'$ the vector

$$L_j = -\sum_{c \leq j} e_c + \sum_{d \geq k-j+1} e_d,$$

where $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{R}^k$. We know that the set $\Lambda(\mathcal{E}, \phi)$ is the positive span of the vectors $\{L_j \mid j \in \mathcal{J}'\}$. Consequently, we have

$$d(\mathcal{E}, \lambda) \geq 0 \text{ for any } \lambda \in \Lambda(\mathcal{E}, \phi) \ \Leftrightarrow d(\mathcal{E}, L_j) \geq 0 \text{ for any } j$$

and $d(\mathcal{E}, L_j) = -\deg E_{k-j} - \deg E_j$. Since $\deg E_{k-j} = \deg E_j$, $d(\mathcal{E}, L_j) \geq 0$ is equivalent to $\deg E_j \leq 0$, which holds by assumption. Hence $(\mathbf{E}, Q, \phi)$ is semistable.

Conversely, if $(\mathbf{E}, Q, \phi)$ is semistable then for any isotropic subbundle $E' \subset \mathbf{E}$ such that $\phi(E') \subseteq E' \otimes K$ we have $\deg E' \leq 0$ is immediate by applying the semistability condition of the filtration $0 \subset E' \subset E'^{\perp_Q} \subset \mathbf{E}$.

Finally, the proof of the second statement on stability is very similar to the case of semistability, we thus omit it. $\square$

Proposition 2.7. *Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle and let $(\boldsymbol{E}, Q, \phi)$ be the corresponding $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle. If $(\mathcal{O}, W, Q_W, \eta)$ is stable, then $(\boldsymbol{E}, Q, \phi)$ is stable as $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle.*

Proof. Let $(\mathcal{O}, W, Q_W, \eta)$ be a semistable $\mathrm{SO}_0(1,n)$-Higgs bundle and consider the associated $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle $(\mathbf{E}, Q, \phi)$. We will see that for every isotropic subbundle $E' \subset \mathbf{E}$ such that $\phi(E') \subseteq E'$ we have $\deg E' \leq 0$.

If $E' \subset \mathbf{E}$ is an isotropic subbundle, we consider the projection $p : \mathbf{E} \to W$ and the subbundles $W' = p(E')$ and $V' = E' \cap \mathcal{O}$. Observe that $V' = \mathcal{O}$ or 0. We have the exact sequence

$$0 \to V' \to E' \to W' \to 0$$

and the equality

$$\deg E' = \deg W'.$$

Since $Q = \begin{pmatrix} 1 & \\ & Q_W \end{pmatrix}$, we have

$$\begin{aligned} (E')^{\perp_{\mathbf{E}}} &= (V' \oplus W')^{\perp_{\mathbf{E}}} = (V')^{\perp_{\mathbf{E}}} \cap (W')^{\perp_{\mathbf{E}}} \\ &= [(V')^{\perp_{\mathcal{O}}} \oplus W] \cap [V \oplus (W')^{\perp_W}] = (V')^{\perp_{\mathcal{O}}} \oplus (W')^{\perp_W}, \end{aligned}$$

and then, the condition $E' \subseteq (E')^{\perp_{\mathbf{E}}}$ implies $V' \subseteq (V')^{\perp_{\mathcal{O}}}$ and $W' \subseteq (W')^{\perp_W}$, that is, V' and W' are isotropic subbundles of $\mathcal{O}$ and W respectively. This implies that $V' = 0$. On the other hand, since $\phi(E') \subseteq E' \otimes K$ and $\phi = \begin{pmatrix} & \eta \\ -\eta^\top & \end{pmatrix}$, we have that $\eta(W') = 0$.

The semistability condition for $(\mathcal{O}, W, Q_W, \eta)$ gives $\deg E' = \deg W' \leq 0$ and then we conclude that the semistability of an $\mathrm{SO}_0(1,n)$-Higgs bundle implies the semistability of its associated $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle.

Let now $E' \subset \mathbf{E}$ be a non-zero isotropic subbundle such that $\phi(E') \subseteq E' \otimes K$. Since $E' \neq 0$ and it is isotropic, $W' = p(E')$ is non-zero. The stability condition for $(\mathcal{O}, W, Q_W, \eta)$ gives $\deg E' = \deg W' < 0$ and we conclude. □

3. Polystable $\mathrm{SO}_0(1,2m+1)$-Higgs bundles

The main result in this section is Theorem 3.1 which gives a full description of polystable $\mathrm{SO}_0(1,n)$-Higgs bundles.

For this, we need to describe some special $\mathrm{SO}_0(1,n)$-Higgs bundles which arise from G-Higgs bundles, for certain real subgroups G of $\mathrm{SO}_0(1,n)$. Consider $\mathrm{U}(n') \subset \mathrm{SO}_0(1,n)$. From a $\mathrm{U}(n')$-Higgs bundle, that is, a holomorphic vector bundle W' of rank n', we can obtain an $\mathrm{SO}(2n')$-Higgs bundle considering the orthogonal bundle $(W' \oplus W'^*, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ denotes the dual pairing. This principal $\mathrm{SO}(2n',\mathbb{C})$-bundle is a special $\mathrm{SO}_0(1,n)$-Higgs bundle with $V = 0$, $(W,Q) = (W' \oplus W'^*, \langle\cdot,\cdot\rangle)$, $n = 2n'$ and $\eta = 0$. Consider now the inclusion $\mathrm{SO}(n') \subset \mathrm{SO}_0(1,n)$. An $\mathrm{SO}(n')$-Higgs bundle (W',Q') corresponds to an $\mathrm{SO}_0(1,n)$-Higgs bundle with $V = 0$, $(W,Q) = (W',Q')$, $n = n'$ and $\eta = 0$.

Theorem 3.1. *Let $(\mathcal{O}, W, Q_W, \eta)$ be a polystable $\mathrm{SO}_0(1,n)$-Higgs bundle. There is a decomposition, unique up to reordering, of this Higgs bundle as a sum of stable G_i-Higgs bundles, where G_i is one of the following groups: $\mathrm{SO}_0(1,n_i)$, $\mathrm{SO}(n_i)$ or $\mathrm{U}(n_i)$.*

Proof. Let $(\mathcal{O}, W, Q_W, \eta)$ be a polystable $\mathrm{SO}_0(1,n)$-Higgs bundle. For (W, Q_W) we fix a filtration $\mathcal{W} = (0 \subset W_1 \subset \cdots \subset W_s = W)$ with $W_j = W_{s-j}^{\perp_{Q_W}}$ and a strictly antidominant character $\mu_1 < \ldots < \mu_s$ with $\mu_{s-i+1} + \mu_i = 0$, such that $\eta \in H^0(\bigoplus_{\mu_i \geq 0} \mathrm{Hom}(W_i, \mathcal{O}) \otimes K)$ and $d(\mathcal{W}, \mu) = 0$. Since $(\mathcal{O}, W, Q_W, \eta)$ is polystable, we have

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W/W_{s-1},$$

with

$$Q_W(W_i/W_{i-1}, W_j/W_{j-1}) = 0 \text{ unless } i + j = s + 1,$$

and

$$\eta \in H^0(\bigoplus_{\mu_i = 0} \mathrm{Hom}(W_i/W_{i-1}, \mathcal{O}) \otimes K).$$

From the condition

$$Q_W(W_i/W_{i-1}, W_j/W_{j-1}) = 0 \text{ unless } i+j = s+1,$$

we have that the bilinear form Q_W gives an isomorphism $(W_i/W_{i-1})^* \cong W_{s-i+1}/W_{s-i}$. We have the exact sequence

$$W_i^\perp \longrightarrow W_{i-1}^\perp \xrightarrow{p} (W_i/W_{i-1})^*$$

where p is given by $w \mapsto Q_W(W, \cdot)$, and then

$$(W_i/W_{i-1})^* \cong W_{i-1}^\perp/W_i^\perp \cong W_{s-i+1}/W_{s-i}.$$

Suppose that n is odd and that we have a filtration $\mathcal{W} = (0 \subset W_1 \subset ... \subset W_s = W)$ where s is even, then $W_{\frac{s}{2}}^\perp = W_{s-\frac{s}{2}} = W_{\frac{s}{2}}$. On the other hand, $\mathrm{rk}(W_{\frac{s}{2}}^\perp) = n - \mathrm{rk}(W_{\frac{s}{2}})$, that implies $\mathrm{rk}(W_{\frac{s}{2}}) = \frac{n}{2}$, which is not a natural number. Then, if n is odd, all the possible filtrations $\mathcal{W} = (0 \subset W_1 \subset ... \subset W_s = W)$, have odd length s, and the value 0 always appears in the middle of the corresponding strictly antidominant character $\mu_1 < ... < \mu_s$. When the rank n is even, we have filtrations for all $1 \leq s \leq n$. When s is odd, we have $\mu_{\frac{s+1}{2}} = 0$ and in the even case, we have $\cdots \mu_{\frac{s}{2}} < \mu_{\frac{s}{2}+1} \cdots$, with $\mu_{\frac{s}{2}} = -\mu_{\frac{s}{2}+1} < 0$.

The Higgs field can be not equal to zero only when $\mu_{\frac{s+1}{2}} = 0$, that is

$$\eta \in H^0(\mathrm{Hom}(W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}}, \mathcal{O}) \otimes K).$$

Since

$$(W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}})^* \cong W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}},$$

the tuple

$$(\mathcal{O},\, W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}},\, Q_W,\, \eta)$$

is in itself an $\mathrm{SO}_0(1, n_i)$-Higgs bundle, where $n_i = \mathrm{rk}(W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}})$. Observe that Q_W denotes now the restriction to $W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}}$.

If $0 = \eta \in H^0(\mathrm{Hom}(W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}}, \mathcal{O}) \otimes K)$, $(\mathcal{O}, W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}}, Q_W, \eta)$ is the sum of the trivial bundle together with an $\mathrm{SO}(n_i)$-Higgs bundle $(W_{\frac{s+1}{2}}/W_{\frac{s-1}{2}}, Q_W)$.

When $\mu_i \neq 0$, we have a pair of $\mathrm{U}(n_i)$-Higgs bundles

$$W_i/W_{i-1} \text{ and } W_{s-i+1}/W_{s-i},$$

dual one to the other. In this case $n_i = \mathrm{rk}(W_i/W_{i-1}) = \mathrm{rk}(W_{s-i+1}/W_{s-i})$.

Each piece in the decomposition is also polystable, and we can repeat the process and obtain a decomposition where all the pieces are stable Higgs bundles (using the Jordan-Hölder reduction, [8, Sec. 2.10]). □

Observe that there can only be one summand with $G_i = \mathrm{SO}_0(1, n_i)$ in the decomposition.

If in the decomposition of a polystable $\mathrm{SO}_0(1,n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ there is a summand which is an $\mathrm{SO}(2)$-Higgs bundle, that is, a principal $\mathrm{SO}(2,\mathbb{C})$-bundle $E = L \oplus L^{-1}$, the isotropic subbundles L and L^{-1}, which have opposite degrees, do not violate the stability condition for E (since there are no parabolic subgroups in $\mathrm{SO}(2,\mathbb{C})$) but they violate the stability condition for $(\mathcal{O}, W, Q_W, \eta)$.

Equivalently, if there is a summand in the decomposition which is a $\mathrm{U}(n_i)$-Higgs bundles E, then E^* is also in the decomposition of $(\mathcal{O}, W, Q_W, \eta)$, and since $\deg(E) = -\deg(E)$, one or both vector bundles violate the stability condition for $(\mathcal{O}, W, Q_W, \eta)$.

Then we have the following results.

Proposition 3.2. *If a polystable* $\mathrm{SO}_0(1,n)$*-Higgs bundle* $(\mathcal{O}, W, Q_W, \eta)$ *decomposes as a sum of stable* G_i*-Higgs bundles where* $G_i = \mathrm{SO}_0(1,n_i)$ *and* $\mathrm{SO}(n_i)$ *with* $n_i \neq 2$*, then* $(\mathcal{O}, W, Q_W, \eta)$ *is stable.*

Proposition 3.3. *If an* $\mathrm{SO}_0(1,n)$*-Higgs bundle* $(\mathcal{O}, W, Q_W, \eta)$ *is strictly polystable, then in its decomposition there must be at least a* G_i*-Higgs bundle with* $G_i = \mathrm{U}(n_i)$ *or* $\mathrm{SO}(2)$*.*

Theorem 3.1 gives us a decomposition of a polystable $\mathrm{SO}_0(1,n)$-Higgs bundles as a sum of stable G_i-Higgs bundles, where G_i is one of the following groups: $\mathrm{SO}_0(1,n_i)$, $\mathrm{SO}(n_i)$ or $\mathrm{U}(n_i)$. From the following result we have that, in fact, any polystable $\mathrm{SO}_0(1,n)$-Higgs bundles can be decomposed as a sum of smooth G_i-Higgs bundles.

Proposition 3.4. *Let* $(\mathcal{O}, W, Q_W, \eta)$ *be a polystable* $\mathrm{SO}_0(1,n)$*-Higgs bundle. There is a decomposition, unique up to reordering, of this Higgs bundle in a sum of smooth* G_i*-Higgs bundles, where* $G_i = \mathrm{SO}_0(1,n_i)$*,* $\mathrm{SO}(n_i)$ *or* $\mathrm{U}(n_i)$*.*

Proof. The starting point is Theorem 3.1.

A stable $\mathrm{U}(n)$-Higgs bundle represents a smooth point in the moduli space of $\mathrm{U}(n)$-Higgs bundles.

A stable $\mathrm{SO}(n)$-Higgs bundle is smooth if and only if it is stable and simple. On the other hand, any stable $\mathrm{SO}(n)$-Higgs bundle which is not simple can be expressed, using Theorem 5.2, as a direct sum of smooth $\mathrm{SO}(n_i)$-Higgs bundles.

Finally, as we know from Corollary 4.4, a stable $\mathrm{SO}_0(1,n)$-Higgs bundle represents a smooth point of the moduli space if and only if it is simple, but if a stable $\mathrm{SO}_0(1,n)$-Higgs bundle is non-simple, from Theorem 5.5 we have that it decomposes as a sum of smooth $\mathrm{SO}_0(1,n_i)$ and $\mathrm{SO}(n_i)$-Higgs bundles. ☐

4. Smoothness and deformation theory

It is known that a stable vector bundle is simple and that it is a smooth point of the moduli space of polystable vector bundles. On the other hand, a stable principal $\mathrm{SO}(n,\mathbb{C})$-bundle with $n \neq 2$ represents a smooth point of the moduli space $\mathcal{M}(\mathrm{SO}(n))$ if and only if it is simple (see [16]). Observe that, for $n = 2$, we have $\mathrm{SO}(2,\mathbb{C}) \cong \mathbb{C}^*$ and then any $\mathrm{SO}(2)$-Higgs bundle is stable, simple and smooth. Thus, except in the case $n = 2$, the stability of a special orthogonal bundle does not imply simplicity. In this section we study the smoothness conditions in the moduli space $\mathcal{M}(\mathrm{SO}_0(1,n))$ adapting the results in [8, Sec. 4.2] to our case.

Definition 4.1. A G-Higgs bundle (E,φ) is said to be **simple** if $\mathrm{Aut}(E,\varphi) = \ker \iota \cap Z(H^{\mathbb{C}})$, where $H \subset G$ is a maximal compact subgroup, $Z(H^{\mathbb{C}})$ denotes the centre of its complexification and $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$ is the complexified isotropy representation corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra of G.

A G-Higgs bundle is then simple if the group of automorphisms is as small as possible. To be in $\ker \iota$ means to be compatible with the Higgs field.

If (E, Q) is an $\mathrm{SO}(n)$-Higgs bundle with $n > 2$, that is, a principal $\mathrm{SO}(n, \mathbb{C})$-bundle, it has Higgs field equal to zero and then it is simple if and only if

$$\mathrm{Aut}(E, Q) = Z(\mathrm{SO}(n, \mathbb{C})) = \begin{cases} I_n, & n \text{ odd}, \\ \pm I_n, & n \text{ even}. \end{cases}$$

The group of automorphisms of an $\mathrm{SO}_0(1, n)$-Higgs bundle is

$$\mathrm{Aut}(\mathcal{O}, W, Q_W, \eta) = \{(1, g) \in \mathrm{Aut}(\mathcal{O}) \times \mathrm{Aut}(W, Q_W) \mid \eta \circ g = \eta\},$$

and hence $(\mathcal{O}, W, Q_W, \eta)$ is simple if and only if

$$\mathrm{Aut}(\mathcal{O}, W, Q_W, \eta) = \ker \iota \cap \{1\} \times Z(\mathrm{SO}(n, \mathbb{C})) = \{I_{n+1}\}.$$

Let us consider the **deformation complex** of an $\mathrm{SO}_0(1, n)$-Higgs bundle

$$\begin{array}{rcl} C^\bullet(\mathcal{O}, W, Q_W, \eta) : \mathfrak{so}(\mathcal{O}) \oplus \mathfrak{so}(W) & \to & \mathrm{Hom}(W, \mathcal{O}) \otimes K, \\ (0, g) & \mapsto & \eta g. \end{array}$$

(see [8, Definition 4.3]). We have the following result.

Proposition 4.2. *If $(\mathcal{O}, W, Q_W, \eta)$ is an $\mathrm{SO}_0(1, n)$-Higgs bundle, we have the following:*

(1) The space of endomorphisms of $(\mathcal{O}, W, Q_W, \eta)$ is isomorphic to the hypercohomology group $\mathbb{H}^0(C^\bullet(\mathcal{O}, W, Q_W, \eta))$.

(2) The space of infinitesimal deformations of $(\mathcal{O}, W, Q_W, \eta)$ is isomorphic to the first hypercohomology group $\mathbb{H}^1(C^\bullet(\mathcal{O}, W, Q_W, \eta))$.

It follows from Proposition 4.2 that, for every $\mathrm{SO}_0(1, n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ representing a smooth point of the moduli space, the tangent space at this point is canonically isomorphic to $\mathbb{H}^1(C^\bullet(\mathcal{O}, W, Q_W, \eta))$.

Proposition 4.3. *If an $\mathrm{SO}_0(1, n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ is stable, simple and satisfies*

$$\mathbb{H}^2(C^\bullet(\mathcal{O}, W, Q_W, \eta)) = 0,$$

then it is a smooth point of the moduli space.

A G-Higgs bundle (E, φ) is **infinitesimally simple** if $\mathrm{End}(E, \varphi) \cong \mathbb{H}^0(C^\bullet(E, \varphi))$ is isomorphic to $H^0(E(\ker d\iota \cap \mathfrak{z}))$. Stable implies infinitesimally simple.

Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1, n)$-Higgs bundle and consider the associated $\mathrm{SO}(n+1, \mathbb{C})$-Higgs bundle $(\mathbf{E}, Q, \phi)$ and the deformation complex

$$C^\bullet(\mathbf{E}, Q, \phi) : \mathfrak{so}(\mathbf{E}) \xrightarrow{\mathrm{ad}(\varphi)} \mathfrak{so}(\mathbf{E}) \otimes K.$$

Since $\mathrm{SO}(n+1, \mathbb{C})$ is complex, infinitesimally simple in this case means $\mathbb{H}^0(C^\bullet(\mathbf{E}, Q, \phi)) = 0$ ($\ker d\iota = \ker(\mathrm{ad}) = 0$) and, as in the real case, stable implies infinitesimally simple. There is an isomorphism

$$\mathbb{H}^2(C^\bullet(\mathbf{E}, Q, \phi)) = \mathbb{H}^0(C^\bullet(\mathbf{E}, Q, \phi))^*,$$

and we have the following relation

$$\mathbb{H}^0(C^\bullet(\mathbf{E}, Q, \phi)) \cong \mathbb{H}^0(C^\bullet(E, \varphi)) \oplus \mathbb{H}^2(C^\bullet(E, \varphi))^*.$$

Then, if $(\mathbf{E}, Q, \phi)$ is stable, $\mathbb{H}^0(C^\bullet(\mathbf{E}, Q, \phi)) = 0$ and this implies

$$\mathbb{H}^0(C^\bullet(E, \varphi)) = \mathbb{H}^2(C^\bullet(E, \varphi)) = 0.$$

Using Proposition 2.7 we obtain the following description.

Corollary 4.4. *If an* $\mathrm{SO}_0(1,n)$*-Higgs bundle* $(\mathcal{O}, W, Q_W, \eta)$ *is stable and simple, then it is a smooth point of the moduli space.*

Corollary 4.5. *Let* $(\mathcal{O}, W, Q_W, \eta)$ *be a stable* $\mathrm{SO}_0(1,n)$*-Higgs bundle which represents a smooth point of the moduli space, then*

$$\mathbb{H}^0(C^\bullet(\mathcal{O}, W, Q_W, \eta)) = \mathbb{H}^2(C^\bullet(\mathcal{O}, W, Q_W, \eta)) = 0.$$

The **expected dimension** of the moduli space $\mathcal{M}(\mathrm{SO}_0(1,n))$ (see [8]), is

$$\dim \mathbb{H}^1(C^\bullet(\mathcal{O}, W, Q_W, \eta)) = -\chi(C^\bullet(\mathcal{O}, W, Q_W, \eta)) = \frac{n(n+1)(g-1)}{2},$$

where $\dim(\mathrm{SO}_0(1,n)) = \frac{n(n+1)}{2}$.

5. Stable and non-simple $\mathrm{SO}(n)$ and $\mathrm{SO}_0(1,n)$-Higgs bundles

In this section we give a description of the stable $\mathrm{SO}(n)$ and $\mathrm{SO}_0(1,n)$-Higgs bundles that fail to be simple.

Lemma 5.1. *If an* $\mathrm{SO}(n)$*-Higgs bundle* (E, Q) *decomposes as a sum of* G_i*-Higgs bundles and one of them is an* $\mathrm{SO}(n_i)$*-Higgs bundle with* $n_i > 2$ *which is not stable or an* $\mathrm{SO}(2)$*-Higgs bundle, then* (E, Q) *is not stable.*

Proof. If there is a summand which is an $\mathrm{SO}(2)$-Higgs bundle $E_i = L \oplus L^{-1}$, the isotropic subbundles L and L^{-1}, which have opposite degrees, do not violate the stability condition for E but they violate the stability condition for (E, Q). If a summand (E_i, Q_i) is a non-stable $\mathrm{SO}(n_i)$-Higgs bundle, there is a proper isotropic subbundle $F_i \subset E_i$ such that $\deg F_i \geq 0$. Since Q_i is the restriction of Q to E_i, F_i is an isotropic subbundle of E that violates its stability. □

Theorem 5.2. *Let* (E, Q) *be a stable* $\mathrm{SO}(n)$*-Higgs bundle with* $n \neq 2$*, that is, a principal* $\mathrm{SO}(n, \mathbb{C})$*-bundle, which is not simple, then it decomposes as a sum of stable and simple* $\mathrm{SO}(n_i)$*-Higgs bundles with* $n_i \neq 2$*. Moreover, in the decomposition there must be at least an* $\mathrm{SO}(n_i)$*-Higgs bundle with* n_i *odd.*

Proof. Since (E, Q) is not simple and

$$Z(\mathrm{SO}(n, \mathbb{C})) = \begin{cases} I_n, & n \text{ odd}, \\ \pm I_n, & n \text{ even}, \end{cases}$$

there is an automorphism $f \in \mathrm{Aut}(E, Q) \setminus \{\pm I_n\}$ if n even, or $f \in \mathrm{Aut}(E, \varphi) \setminus \{I_n\}$ if n is odd.

Suppose that $f = \lambda I_n$ with $\lambda \in \mathbb{C}^*$. It has to preserve the orthogonal structure of E, that is,

$$Q(f(e), f(e')) = \lambda^2 Q(e, e') = Q(e, e'),$$

and this happens if and only if $\lambda = \pm 1$. On the other hand, the determinant of f has to be equal to one. Then, the only possibilities are $f = \pm I_n$ if n is even and $f = I_n$ if n is odd, which are exactly the cases that we are excluding.

The group $\mathrm{Aut}(E,\varphi)$ is reductive. This implies that f may be chosen in such a way that there is a splitting $E = \bigoplus E_i$ such that f restricted to E_i is $\lambda_i I_n$ with $\lambda_i \in \mathbb{C}^*$.

Since

$$Q(e_i, e_j) = Q(f(e_i), f(e_j)) = \lambda_i \lambda_j Q(e_i, e_j),$$

then $Q(E_i, E_j)$ can only be non-zero when $\lambda_i\lambda_j = 1$. Since Q is non-degenerate, the possible values of lambda come in pairs $(\lambda_i, \lambda_i^{-1})$ corresponding to (E_i, E_i^*). If $\lambda_i = \pm 1$, we have $\lambda_i = \lambda_i^{-1}$ and then $E_1 \cong E_1^*$ and $E_{-1} \cong E_{-1}^*$. Since $\det f = \prod_i \lambda_i^{\mathrm{rk}\, E_i} = 1$, we do not have the value $\lambda_i = 0$.

Suppose that there is a $\lambda_i \neq \pm 1$, then $E_i \subset E$ is an isotropic subbundle of E. If $\deg E_i \geq 0$, this subbundle violates the stability condition for (E,Q). If $\deg E_i < 0$, then $\deg E_i^* > 0$ and again (E,Q) is not stable. Hence $\lambda_i = \pm 1$ and $(E,Q) = (E_1, Q_1) \oplus (E_{-1}, Q_{-1})$.

From Lemma 5.1 we have that these summands are stable $\mathrm{SO}(n_i)$-Higgs bundles with $n_i \neq 2$.

If there is a summand which is a non-simple $\mathrm{SO}(n_i)$-Higgs bundle, applying the argument of this proof inductively we conclude that a stable but non-simple $\mathrm{SO}(n)$-Higgs bundle can be decomposed as a sum of smooth $\mathrm{SO}(n_i)$-Higgs bundles.

Finally, since (E,Q) is not simple, there must be at least an $\mathrm{SO}(n_i)$-Higgs bundle with n_i even in the decomposition. This condition allows us to take the automorphism -1 in this summand and guarantee the non-simplicity. □

Lemma 5.3. *If an $\mathrm{SO}_0(1,n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ decomposes as a sum of G_i-Higgs bundles and one of them is an $\mathrm{SO}_0(1,n_i)$-Higgs bundle $(\mathcal{O}, W_i, Q_W, \eta_i)$ which is not stable, then $(\mathcal{O}, W, Q_W, \eta)$ is not stable.*

Proof. Since $(\mathcal{O}, W_i, Q_W, \eta_i)$ is not stable, there is an isotropic subbundle $W' \subset W_i$ (such that $\eta_i(W') \subseteq \mathcal{O} \otimes K$) with $\deg W' \geq 0$. But W' is also an isotropic subbundle of W and violates the stability condition for $(\mathcal{O}, W, Q_W, \eta)$. □

Lemma 5.4. *If an $\mathrm{SO}_0(1,n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ decomposes as a sum of G_i-Higgs bundles and one of them is an $\mathrm{SO}(2)$-Higgs bundle or an $\mathrm{SO}(n_i)$-Higgs bundle which is not stable, then $(\mathcal{O}, W, Q_W, \eta)$ is not stable.*

Proof. It can be deduced from the proof of Lemma 5.1 and Lemma 5.3. □

Theorem 5.5. *Let $(\mathcal{O}, W, Q_W, \eta)$ be a stable $\mathrm{SO}_0(1,n)$-Higgs bundle which is not simple, then it decomposes as a sum of stable and simple $\mathrm{SO}_0(1,n_i)$-Higgs bundles and stable and simple $\mathrm{SO}(n_i)$-Higgs bundles with $n_i \neq 2$. Moreover, in the decomposition there must be at least an $\mathrm{SO}(n_i)$-Higgs bundle with n_i even.*

Proof. Suppose that the Higgs field is equal to zero, then the $\mathrm{SO}_0(1,n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ is the sum of the trivial bundle together with a stable principal $\mathrm{SO}(n,\mathbb{C})$-bundle (W, Q_W), that is, a stable $\mathrm{SO}(n)$-Higgs bundle. If (W, Q_W) is simple, then we have the result. If it is not, we conclude using Theorem 5.2.

Suppose now that $\eta \neq 0$. Since $(\mathcal{O}, W, Q_W, \eta)$ is not simple, there is an automorphism $f \in \mathrm{Aut}(\mathcal{O}, W, Q_W, \eta) \backslash \{I\}$. If $f = (f_1, f_2)$, since $f_1 \in \mathrm{Aut}(\mathcal{O})$, we have $f_1 = 1$.

Suppose that $f = (f_1, f_2) = (1, \mu I)$ is a multiple of the identity in W ($\mu \in \mathbb{C}^*$). The determinant of f_2 has to be equal to 1 and f_2 has to preserve the orthogonal structure, that

is,

$$Q_W(f_2(w), f_2(w')) = \mu^2 Q_W(w, w') = Q_W(w, w').$$

On the other hand, since we are supposing that f_2 is a multiple of the identity, the condition $f_1 \circ \eta = \eta \circ f_2$ is equivalent to $f_1 = f_2$, that is $f = I$, which is exactly the case that we are excluding. Thus, f is not of this form.

Since the group $\mathrm{Aut}(W, Q_W)$ is reductive, there is a splitting $W = \bigoplus W_i$ such that $f_2 = \mu_i I$ in W_i ($\mu_i \in \mathbb{C}^*$). Since

$$Q_W(w_i, w_j) = Q_W(f_2(w_i), f_2(w_j)) = \mu_i \mu_j Q_W(w_i, w_j),$$

then $Q_W(W_i, W_j)$ can only be non-zero when $\mu_i \mu_j = 1$. Since Q_W is non-degenerate, the possible values of mu come in pairs (μ_i, μ_i^{-1}) corresponding to (W_i, W_i^*). If $\mu_i = \pm 1$, we have $\mu_i = \mu_i^{-1}$ and then $W_1 \cong W_1^*$ and $W_{-1} \cong W_{-1}^*$. Since $\det f_2 = \prod_i \mu_i^{\mathrm{rk}\, W_i} = 1$, we do not have $\mu_i = 0$.

Since f preserve the Higgs field, for each component $\eta_i \in H^0(\mathrm{Hom}(W_i, \mathcal{O}) \otimes K)$, we have that

$$\eta_i(f_2(w)) = \mu_i \eta_i(w)$$

is equal to

$$f_1(\eta_i(w)) = \eta_i(w),$$

for all $w \in W_i$, and then, $\mu_i \neq 1$ implies $\eta_i = 0$.

Suppose that there is a $\mu_i \neq \pm 1$. Then, in particular, $\mu_i \neq 1$ and we have $\eta_i = 0$, that is, $\eta(W_i) = 0$. Since

$$Q_W(W_i, W_i) = Q_W(f_2(W_i), f_2(W_i)) = \mu_i^2 Q_W(W_i, W_i),$$

and $\mu_i^2 \neq 1$, we have $Q_W(W_i, W_i) = 0$ and hence, $W_i \subset W$ is an isotropic subbundle. If $\deg W_i \geq 0$, this subbundle violates the stability condition for $(\mathcal{O}, W, Q_W, \eta)$. If $\deg W_i < 0$, then $\deg W_i^* > 0$ and again $(\mathcal{O}, W, Q_W, \eta)$ is not stable and we get a contradiction. Then $\mu_i = \pm 1$.

Since $1 = \det f_2 = 1^{\mathrm{rk}\, W_1} \cdot (-1)^{\mathrm{rk}\, W_{-1}}$ we have $\mathrm{rk}\, W_{-1}$ even.

We have the following decomposition

$$(\mathcal{O}, W, Q_W, \eta) = (\mathcal{O}, W_1, \eta_1) \oplus W_{-1}.$$

Since f_2 is not a multiple of the identity, W_1 and W_{-1} are non-zero, and since $\eta \neq 0$, then $\eta_1 \neq 0$. Thus, $(\mathcal{O}, W, Q_W, \eta)$ is a sum of a $\mathrm{SO}_0(1, n_i)$-Higgs bundle $(\mathcal{O}, W_1, \eta_1)$ together with an $\mathrm{SO}(n_i)$-Higgs bundle W_{-1}.

From Lemma 5.3 and Lemma 5.4 we have that these summands are stable G_i-Higgs bundles ($\mathrm{SO}(n_i)$ with $n_i \neq 2$).

If W_{-1} is non-simple, we have from Theorem 5.2 that it decomposes as a sum of stable and simple orthogonal bundles. If $(\mathcal{O}, W_1, \eta_1)$ is a non-simple $\mathrm{SO}_0(1, n_i)$-Higgs bundle, applying the argument of this proof inductively we conclude that it can be decomposed as a sum of stable and simple G_i-Higgs bundles with $G_i = \mathrm{SO}_0(1, n_i)$ and $\mathrm{SO}(n_i)$.

Since all the summands are simple and $(\mathcal{O}, W, Q_W, \eta)$ is not simple, it must have at least one summand of this type: a smooth $\mathrm{SO}(n_i)$-Higgs bundle with n_i even. This condition allow us to take the automorphism -1 in this summand and guarantee the non-simplicity. $\square$

6. Topology of the moduli spaces

Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle. We have a topological invariant c associated to it, which is given by the following exact sequence

$$1 \to \pi_1(\mathrm{SO}(n,\mathbb{C})) \to \widetilde{\mathrm{SO}}(n,\mathbb{C}) \to \mathrm{SO}(n,\mathbb{C}) \to 1,$$

where $\widetilde{\mathrm{SO}}(n,\mathbb{C})$ is the universal cover of $\mathrm{SO}(n,\mathbb{C})$ and the associated long cohomology sequence

$$H^1(X, \widetilde{\mathrm{SO}}(n,\mathbb{C})) \longrightarrow H^1(X, \mathrm{SO}(n,\mathbb{C})) \xrightarrow{\ c\ } H^2(X, \pi_1(\mathrm{SO}(n,\mathbb{C}))).$$

This invariant

$$c \in H^2(X, \pi_1(\mathrm{SO}(n,\mathbb{C}))) \cong \pi_1(\mathrm{SO}(n,\mathbb{C}))$$

measures the obstruction to lifting (W, Q_W) to a flat $\widetilde{\mathrm{SO}}(n,\mathbb{C})$)-bundle. Observe that when $n \geq 3$, the universal cover of $\mathrm{SO}(n,\mathbb{C})$ is $\mathrm{Spin}(n,\mathbb{C})$. We have that

$$\pi_1(\mathrm{SO}(n,\mathbb{C})) = \begin{cases} 1, & n = 1, \\ \mathbb{Z}, & n = 2, \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

When $n \geq 3$, the invariant $c \in \mathbb{Z}/2$ corresponds to the second Stiefel-Whitney classe of the orthogonal bundle that we obtain from the reduction of the structure group of (W, Q_W) from $\mathrm{SO}(n,\mathbb{C})$ to the real group $\mathrm{SO}(n)$.

Since $\det W = \mathcal{O}$, using the application

$$H^1(X, \mathrm{SO}(n,\mathbb{C})) \xrightarrow{\det} J(X)$$

in the Jacobian of X and the identification

$$H^1(X, \mathbb{Z}_2) \cong J_2(X) = \{L \in J(X) \mid L^2 \cong \mathcal{O}\},$$

the first Stiefel-Whitney classes of the bundle is zero.

We define the moduli space of polystable $\mathrm{SO}_0(1,n)$- Higgs bundles with invariant c as

$$\mathcal{M}_c(\mathrm{SO}_0(1,n)) = \{(\mathcal{O}, W, Q_W, \eta) \in \mathcal{M}(\mathrm{SO}_0(1,n)) \text{ such that } c(W, Q_W) = c\}.$$

The invariant c gives a first decomposition of the moduli space

$$\mathcal{M}(\mathrm{SO}_0(1,n)) = \coprod_c \mathcal{M}_c(\mathrm{SO}_0(1,n)).$$

To obtain the number of connected components it is necessary to distinguish which of these components $\mathcal{M}_c(\mathrm{SO}_0(1,n))$ are connected and which decompose as a union of connected components.

7. Hitchin fuction

To simplify, we denote $\mathcal{M} := \mathcal{M}_c(\mathrm{SO}_0(1,n))$. Morse-theoretic techniques for studying the topology of moduli spaces of Higgs bundles were introduced by Hitchin [12, 14]. In this section we describe briefly Hitchin's method and we begin the study of our particular case.

The moduli space of equivalence classes of reductive representations in a Lie group G is homeomorphic to the moduli space of polystable G-Higgs bundles. The proof of this result involves the moduli space of solutions to the *Hitchin's equations*. It was proved by Hitchin

[12] and by Simpson [19] for a complex Lie group and by Bradlow, García-Prada, Gothen and Mundet i Riera [5, 8] in the real case, that $\mathcal{M}(G)$ is homeomorphic to the moduli space of solutions to the Hitchin's equations, $\mathcal{M}^{Hit}(G)$, which is defined as the space of pairs (A, φ), where A is a connection on a smooth principal H-bundle E_H and $\varphi \in \Omega^{1,0}(E_H(\mathfrak{m}^{\mathbb{C}}))$, satisfying

$$\begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0, \\ \bar{\partial}_A(\varphi) &= 0, \end{aligned}$$

modulo gauge equivalence.

Using the homeomorphism $\mathcal{M}_c^{Hit}(\mathrm{SO}_0(1,n)) \cong \mathcal{M}$, the **Hitchin function** is defined as the positive function

$$f : \mathcal{M} \to \mathbb{R},$$

given by

$$[A, \varphi] \mapsto \|\varphi\|^2 = \int_X |\varphi|^2 d\,\mathrm{vol},$$

where $[\cdot,\cdot]$ denotes the equivalence class in the moduli space $\mathcal{M}_c^{Hit}(\mathrm{SO}_0(1,n))$ and $|\cdot|$ is the harmonic metric that gives the reduction to $\mathrm{SO}(1) \times \mathrm{SO}(n)$. Equivalently, we can define the map over the moduli space of Higgs pairs, for a fixed $(E, \varphi) \in \mathcal{M}$, by using the L^2-norm $\|\cdot\|$ of the metric that solves the Hitchin's equations.

Proposition 7.1. *The function $f([A,\varphi]) = \|\varphi\|^2$ is a proper map.*

The proof of this result was given by Hitchin in [12, Proposition 7.1].

Even if $\mathcal{M}$ is not smooth, as in our case, the fact that f is a proper map gives information about the connected components of $\mathcal{M}$.

Proposition 7.2. *Let $\mathcal{M}' \subseteq \mathcal{M}$ be a closed subspace and let $\mathcal{N}' \subseteq \mathcal{M}'$ be the subspace of local minima of f on $\mathcal{M}'$. If $\mathcal{N}'$ is connected, then $\mathcal{M}'$ is connected.*

This result is in fact more general. The proper function f has a minimum on each connected component of $\mathcal{M}'$, and then the number of connected components of $\mathcal{M}'$ is bounded by the number of connected components of $\mathcal{N}'$. Thus, we are interested in computing the critical points and more precisely the local minima of f.

To study the critical points of the Hitchin function we use the following results (see [12]).

Proposition 7.3. *The restriction of $f([A,\varphi]) = \|\varphi\|^2$ to the smooth locus $\mathcal{M}^s \in \mathcal{M}$ is a moment map for the Hamiltonian circle action*

$$[A, \varphi] \mapsto [A, e^{i\theta}\varphi].$$

Proposition 7.4. *A smooth point of the moduli space $\mathcal{M}$ is a critical point of f if and only if it is a fixed point of the circle action, and the subbundle $\nu^-(\mathcal{M}_l)$ where the Hessian of the Hitchin function is negative definite equals the subbundle of $\nu(\mathcal{M}_l)$ on which the circle acts with negative weights.*

Using Proposition 7.4, the critical points of f are of two types:

(1) The Higgs field $\varphi = 0$.

(2) If $\varphi \neq 0$, $[A,\varphi]$ is a fixed point of the circle action if and only if

$$[A, e^{i\theta}\varphi] = [A,\varphi], \text{ for all } e^{i\theta} \in S^1.$$

Then, there is a 1-parameter family of gauge transformations $g(\theta) = (g_1(\theta), g_2(\theta))$ such that

$$(A, e^{i\theta}\varphi) = g(\theta)\cdot(A,\varphi) = (g(\theta)\cdot A, g(\theta)\cdot\varphi). \tag{7.1}$$

If the family $\{g(\theta) = (g_1(\theta), g_2(\theta))\}$ is generated by an infinitesimal gauge transformation $\psi = (\psi_1, \psi_2)$, we have that

$$g(\theta)\cdot\varphi = \iota(g(\theta))(\varphi) = \mathrm{Ad}(g(\theta))(\varphi) = \exp(\mathrm{ad}(\theta\psi))(\varphi),$$

and taking $\frac{d}{d\theta}|_{\theta=0}$ in the second term of the brackets in (7.1) we obtain

$$\frac{d}{d\theta}(e^{i\theta}\varphi)|_{\theta=0} = i\varphi,$$

and

$$\frac{d}{d\theta}(g(\theta)\cdot\varphi)|_{\theta=0} = \frac{d}{d\theta}\exp(\mathrm{ad}(\theta\psi))(\varphi)|_{\theta=0} = \mathrm{ad}(\psi)(\varphi) = [\psi,\varphi].$$

Then

$$[\psi,\varphi] = i\varphi.$$

Let $A = (A_1, A_2)$. Since $g_1(\theta)$ and $g_2(\theta)$ act on A_1 and A_2 separately, we can consider ψ_1 and ψ_2 generating the action of $\{g_1(\theta)\}$ and $\{g_2(\theta)\}$. The equation (7.1) gives the following condition for the action on the connections

$$g_i(\theta)\cdot A_i = g_i(\theta)\circ A_i \circ g_i(\theta)^{-1} = A_i,$$

or equivalently

$$A_i \circ g_i(\theta) = g_i(\theta)\circ A_i,$$

that is, the automorphism $g_i(\theta)$ is parallel with respect to the connection A_i. Then we have

$$d_{A_i}(\psi_i) = 0.$$

That is, the family $\{g(\theta) = (g_1(\theta), g_2(\theta))\}$ is generated by an infinitesimal gauge transformation $\psi = (\psi_1, \psi_2)$ which is covariantly constant, that is,

$$d_{A_1}(\psi_1) = d_{A_2}(\psi_2) = 0$$

and with

$$[\psi,\varphi] = i\varphi.$$

Proposition 7.5. *An* $\mathrm{SO}_0(1,n)$*-Higgs bundle* $(\mathcal{O}, W, Q_W, \eta) \in \mathcal{M}$ *with* $\eta \neq 0$ *represents a fixed point of the circle action if and only it is a Hodge bundle (complex variation of Hodge structure), that is, if and only if the vector bundles* W *have a decomposition*

$$W = \bigoplus_{r=-s}^{s} W_r,$$

with $W_r \cong (W^*)_{-r}$ *and* $\psi_2|_{W_r} = ir$ *for an infinitesimal gauge transformation* ψ_2*. The only piece of Higgs field non-equal to zero is*

$$\eta : W_{-1} \to \mathcal{O}\otimes K \ \textit{(and } \eta^\top : \mathcal{O} \to W_1 \otimes K\textit{)}.$$

Proof. If $(\mathcal{O}, W, Q_W, \eta)$ represents a smooth point of the moduli space which is a critical point of f, then is is a fixed point of the circle action. The condition $d_{A_2}(\psi_2) = 0$ in the context of Higgs bundles means that the infinitesimal gauge transformation ψ_2 gives a decomposition

$$W = \bigoplus_r W_r,$$

where $\underline{r \in \mathbb{R}}$ and $\psi_2|_{W_r} = ir$. Moreover, since ψ_2 is locally in $\mathfrak{so}(n)$, it satisfies $\psi_2 = -\psi_2^\top$. If $q_W : W \cong W^*$ is the isomorphism given by the orthogonal form Q_W, we have $\psi_2^\top = q_W^{-1} \circ \psi_2^t \circ q_W$, and for all $w \in W_r$ we have

$$\psi_2^t(q_W(w)) = q_W(\psi_2^\top(w)) = -q_W(\psi_2(w)) = -irq_W(w),$$

that is,

$$w \in W_r \Leftrightarrow q_W(w) \in (W^*)_{-r}.$$

Hence, we have an isomorphism $W_r \cong (W^*)_{-r}$.

If $w \in W_r$ and $w' \in W_l$,

$$Q_W(\psi_2(w), w') = Q_W(irw, w') = irQ_W(w, w')$$

and, on the other hand,

$$Q_W(\psi_2(w), w') = Q_W(w, \psi_2^\top(w')) = Q_W(w, -\psi_2(w')) = Q_W(w, -ilw') = -ilQ_W(w, w'),$$

that is,

$$i(r + l)Q_W(w, w') = 0.$$

Then, all the W_l are orthogonal to V_r (including $l = r$) under Q_W except $l = -r$. Since Q_W is non-degenerate,

$$Q_W(w, w') = 0 \text{ for all } w' \in W \Rightarrow w = 0,$$

and then, given $0 \neq w \in W_r$, there is a $w' \in W$ with $Q_W(w, w') \neq 0$, that is, a $w' \in W_{-r}$. Then

$$W = \bigoplus_{r=-s}^{s} W_r.$$

We also know that the endomorphism ψ_2 is trace free, then

$$0 = \mathrm{Tr}(\psi_2) = i \sum_{r=-s}^{s} r \,\mathrm{rk}(W_r) \Leftrightarrow \sum_{r=-s}^{s} r \,\mathrm{rk}(W_r) = 0.$$

The condition $[\psi, \varphi] = i\varphi$ for the solution (A, φ) is equivalent in this context to

$$-\eta\psi_2 = i\eta.$$

If $w \in W_r$, we have

$$-\eta(\psi_2(w)) = -\eta(irw) = -ir\eta(w) = i\eta(w) \Leftrightarrow r = -1 \ (\eta \neq 0),$$

and we conclude. □

From Theorem 7.5 together with Proposition 7.4 we have that if $(\mathcal{O}, W, Q_W, \eta)$ is an $\mathrm{SO}_0(1, n)$-Higgs bundle which represents a smooth point of the moduli space, it is a critical point of the Hitchin function if and only if it is a Hodge bundle, but observe that not every Hodge bundle represents a smooth point.

8. Smooth minima

In this section we study the smooth minima of the Hitchin function in the moduli space of $\mathrm{SO}_0(1,n)$-Higgs bundles.

Let (E,φ) be an $\mathrm{SO}_0(1,n)$-Higgs bundle and let $(E_{\mathrm{SO}(n+1,\mathbb{C})},\varphi)$ be the associated $\mathrm{SO}(n+1,\mathbb{C})$-Higgs bundle. Consider also the tuple $(\mathcal{O},W,Q_W,\eta)$ corresponding to (E,φ) and the triple $(\mathbf{E},Q,\varphi)$ corresponding to $(E_{\mathrm{SO}(n+1,\mathbb{C})},\varphi)$. We have that

$$\begin{aligned} E_{\mathrm{SO}(n+1,\mathbb{C})}(\mathfrak{so}(n+1,\mathbb{C})) &= \{f\in\mathrm{End}(\mathbf{E}) \mid f+f^\top=0\}=\mathfrak{so}(\mathbf{E}),\\ E(\mathfrak{h}^{\mathbb{C}}) &= \{\begin{pmatrix} 0 & 0\\ 0 & f_4\end{pmatrix}\in\mathrm{End}(\mathbf{E}) \mid f_4+f_4^\top=0\}\\ &\cong \mathfrak{so}(\mathcal{O})\oplus\mathfrak{so}(W)\subset\mathrm{End}(\mathcal{O})\oplus\mathrm{End}(W),\\ E(\mathfrak{m}^{\mathbb{C}}) &= \{\begin{pmatrix} 0 & f_2\\ -f_2^\top & 0\end{pmatrix}\in\mathrm{End}(\mathbf{E})\}\cong\mathrm{Hom}(W,\mathcal{O}). \end{aligned}$$

In fact,

$$E_{\mathrm{SO}(n+1,\mathbb{C})}(\mathfrak{so}(n+1,\mathbb{C}))=E(\mathfrak{h}^{\mathbb{C}})\oplus E(\mathfrak{m}^{\mathbb{C}}),$$

which is induced by the Cartan decomposition of the Lie algebra $\mathfrak{so}(n+1,\mathbb{C})$.

If $(\mathcal{O},W,Q_W,\eta)$ is a Hodge bundle, from Proposition 7.5 we have that there is an infinitesimal gauge transformation ψ_2 such that

$$W=\bigoplus_{r=-s}^{s}W_r,$$

with $W_r\cong(W^*)_{-r}$, $\psi_2|_{W_r}=ir$ and

$$\eta: W_{-1}\to\mathcal{O}\otimes K.$$

This decompositions of W gives decompositions

$$\begin{aligned} \mathrm{End}(W) &= \bigoplus_{k=-2s}^{2s}\Big(\bigoplus_{i-j=k}\mathrm{Hom}(W_j,W_i)\Big),\\ \mathrm{Hom}(W,\mathcal{O}) &= \bigoplus_{k=-s}^{s}\mathrm{Hom}(W_k,\mathcal{O}). \end{aligned}$$

If $g_{k,l}\in\mathrm{Hom}(W_k,W_l)$, using the isomorphism q_W induced by the orthogonal form Q_W we have that the diagram

$$\begin{tikzcd} W_l^* \arrow[r, "g_{k,l}^t"] \arrow[d, "\cong"] & W_k^* \arrow[d, "\cong"] \\ W_{-l} \arrow[r, "g_{k,l}^\top"] & W_{-k}, \end{tikzcd}$$

is commutative, and then, the skew-symmetry in $\mathfrak{so}(W)\subset\mathrm{End}(W)$ is equivalent to the condition $g_{-l,-k}+g_{k,l}^\top=0$, that is, the following sets are related by skew-symmetry

$$\begin{aligned} g_{k,l} &\longleftrightarrow -g_{k,l}^\top,\\ \mathrm{Hom}(W_k,W_l) &\longleftrightarrow \mathrm{Hom}(W_{-l},W_{-k}). \end{aligned}$$

Observe that when $k = l$, the endomorphism and $g_{k,l}$ is skew-symmetric. Analogously, in $E(\mathfrak{m}^{\mathbb{C}})$ we have the relation:

$$\begin{array}{rcl} h_k & \longleftrightarrow & -h_k^{\top}, \\ \mathrm{Hom}(W_k, \mathcal{O}) & \longleftrightarrow & \mathrm{Hom}(\mathcal{O}, W_{-k}). \end{array}$$

Then, the decomposition of W also induce decompositions of $E(\mathfrak{h}^{\mathbb{C}}) \cong \mathfrak{so}(\mathcal{O}) \oplus \mathfrak{so}(W) \cong \mathfrak{so}(W)$ and $E(\mathfrak{m}^{\mathbb{C}}) \cong \mathrm{Hom}(W, \mathcal{O})$, which gives a decomposition of the deformation complex of Section 4:

$$C^{\bullet}(\mathcal{O}, W, Q_W, \eta) : \mathfrak{so}(W) \to \mathrm{Hom}(W, \mathcal{O}) \otimes K,$$

given by

$$C^{\bullet}(\mathcal{O}, W, Q_W, \eta) = \bigoplus_k C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta),$$

where $C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)$ are the subcomplexes

$$C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta) : \mathfrak{so}(W)_k \to \mathrm{Hom}(W, \mathcal{O})_{k+1} \otimes K.$$

This induces a decomposition of the infinitesimal deformation space given by

$$\mathbb{H}^1(C^{\bullet}(\mathcal{O}, W, Q_W, \eta)) = \bigoplus_k \mathbb{H}^1(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)).$$

A convenient reference for the following results is García-Prada, Gothen and Mundet i Riera [8].

Proposition 8.1. *Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle which represents a smooth point of the moduli space $\mathcal{M}$ and which is a critical point of f. The hypercohomology group $\mathbb{H}^1(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta))$ is isomorphic to the eigenspace of the Hessian of f with eigenvalue $-k$. Then, $(\mathcal{O}, W, Q_W, \eta)$ corresponds to a local minimum of f if and only if*

$$\mathbb{H}^1(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)) = 0 \textit{ for } k > 0.$$

To give a criterion for deciding when the hypercohomology $\mathbb{H}(C_k^{\bullet}(E, \varphi))$ vanishes, we use the Euler characteristic of the complex $C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)$. If we denoted by $\mathrm{h}^i(\mathcal{O}, W, Q_W, \eta)$ the dimension of the hypercohomology group $\mathbb{H}^i(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta))$, the Euler characteristic is defined by

$$\chi(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)) = \mathrm{h}^0(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)) - \mathrm{h}^1(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)) + \mathrm{h}^2(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)).$$

Proposition 8.2. *Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle which represents a fixed point under the circle action on $\mathcal{M}$. Then*

$$\chi(C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta)) \leq 0,$$

and equality holds if and only if the map

$$C_k^{\bullet}(\mathcal{O}, W, Q_W, \eta) : \mathfrak{so}(W)_k \to \mathrm{Hom}(W, \mathcal{O})_{k+1} \otimes K$$

is an isomorphism.

If $(\mathcal{O}, W, Q_W, \eta)$ represents a smooth $\mathrm{SO}_0(1,n)$-Higgs bundle, using Corollary 4.5, we have that

$$\mathbb{H}^0(C_k^\bullet(\mathcal{O}, W, Q_W, \eta)) = \mathbb{H}^2(C_k^\bullet(\mathcal{O}, W, Q_W, \eta)) = 0,$$

and then,

$$-\chi(C_k^\bullet(\mathcal{O}, W, Q_W, \eta)) = \mathrm{h}^1(C_k^\bullet(\mathcal{O}, W, Q_W, \eta)),$$

for all k. Applying Proposition 8.1, we have the following criterion for local minima of f.

Proposition 8.3. *Let $(\mathcal{O}, W, Q_W, \eta)$ be an $\mathrm{SO}_0(1,n)$-Higgs bundle which represents a smooth point of $\mathcal{M}$ and which is a critical point of f. Then it represents a local minimum if and only if*

$$C_k^\bullet(\mathcal{O}, W, Q_W, \eta) : \mathfrak{so}(W)_k \to \mathrm{Hom}(W, \mathcal{O})_{k+1} \otimes K$$

is an isomorphism for all $k > 0$.

Applying this criterion we obtain the following result.

Theorem 8.4. *The smooth minima of the Hitchin function in the moduli space of polystable $\mathrm{SO}_0(1,n)$-Higgs bundles with $n > 2$ have zero Higgs field.*

Proof. Let $(\mathcal{O}, W, Q_W, \eta)$ be a smooth point of the moduli space with $\eta \neq 0$ which is a minimum of the Hitchin function. Since it is stable, we know from Theorem 3.1 and Proposition 3.2 and 3.3 that it decomposes as a sum of stable G_i-Higgs bundles where $G_i = \mathrm{SO}_0(1, n_i)$ and $\mathrm{SO}(n_i)$ with $n_i \neq 2$. Since $(\mathcal{O}, W, Q_W, \eta)$ is a critical point, from Proposition 7.5, the $\mathrm{SO}_0(1, n_i)$-Higgs bundle in the decomposition is of the form

$$W_{-1}^i \to \mathcal{O} \to W_1^i,$$

where $n_i = 2\,\mathrm{rk}(W^i)$ (and $0 < \deg(W_{-1}^i) \leq 2g-2$). Observe that, since $\mathcal{O}$ and the other $\mathrm{SO}(n_i)$-Higgs bundles in the decomposition are self-dual, then they have weight 0.

Since $(\mathcal{O}, W, Q_W, \eta)$ is a minimum of the Hitchin function, using Proposition 8.3, the subcomplex

$$C_2^\bullet(\mathcal{O}, W, Q_W, \eta) : \Lambda^2 W_1^i \to 0$$

has to be an isomorphism. Then $\mathrm{rk}(W_1^i) = \mathrm{rk}(W_{-1}^i) = 1$.

Since the Hitchin function is additive with respect to the direct sum and $(\mathcal{O}, W, Q_W, \eta)$ is a minimum, each G_i-Higgs bundle in the decomposition has to be a minimum on the corresponding moduli space $\mathcal{M}(G_i)$ and a minimum as $\mathrm{SO}_0(1, n_i)$-Higgs bundles. Using the criterion of Proposition 8.3, we have that the $\mathrm{SO}_0(1,2)$-Higgs bundle $(\mathcal{O}, W_1^i \oplus W_{-1}^i, \eta)$ is a minimum. The summands corresponding to $\mathrm{SO}(n_i)$-Higgs bundles are minima, because they have Higgs field equal to zero. Consider now the sum of this Higgs bundle together with an $\mathrm{SO}(n_i)$-Higgs bundle (E, Q) in the decomposition of $(\mathcal{O}, W, Q_W, \eta)$. (Since $n > 2$, there is at least one summand of this type). The subcomplex

$$C_1^\bullet(\mathcal{O}, W_1^i \oplus W_{-1}^i \oplus E, \eta) : \mathrm{Hom}(W_{-1}^i, E) \to 0$$

is not an isomorphism and then $(\mathcal{O}, W_1^i \oplus W_{-1}^i \oplus E, \eta)$ is not a minimum. We get a contradiction and we conclude that the Higgs field η has to be equal to zero. $\square$

9. Minima in the whole moduli space

In the previous section we characterize the minima of the Hitchin functional in the smooth locus of the moduli space of $\mathrm{SO}_0(1,n)$-Higgs bundle. In this section we extend the characterization to the whole moduli space for n odd. This allows us to solve the problem of counting the connected components of $\mathcal{M}(\mathrm{SO}_0(1,n))$ with n odd.

Theorem 9.1. *All the minima of the Hitchin function in the moduli space of polystable $\mathrm{SO}_0(1,n)$-Higgs bundles, with n odd, have the Higgs field equal to zero.*

Proof. From Theorem 8.4 we have that the smooth minima of the Hitchin function in the moduli space of polystable $\mathrm{SO}_0(1,n)$-Higgs bundles have zero Higgs field. In particular this is true for n odd.

1. If $(\mathcal{O}, W, Q_W, \eta)$ is a stable but non-simple $\mathrm{SO}_0(1,n)$-Higgs bundle (n odd) with $\eta \neq 0$ which is a fixed point of the circle action, using Theorem 5.5 and Proposition 7.5, we obtain that it decomposes as a sum of a smooth minimum in $\mathcal{M}(\mathrm{SO}_0(1,n_i))$ of the form

$$W^i_{-1} \to \mathcal{O} \to W^i_1,$$

together with a sum of $\mathrm{SO}(n_i)$-Higgs bundles with $n_i \neq 2$ where at least one has rank n_i even. The first summand is necessary to guarantee the condition $\eta \neq 0$ and the condition for the rank n_i to be even determines the non-simplicity of $(\mathcal{O}, W, Q_W, \eta)$.

As in the proof of Theorem 8.4, since the Hitchin function f is additive with respect to the direct sum, if $(\mathcal{O}, W, Q_W, \eta)$ is a minimum, each Higgs bundle in its decomposition has to be a minimum on the corresponding moduli space $\mathcal{M}(G_i)$ and a minimum as $\mathrm{SO}_0(1,n_i)$-Higgs bundle.

Since $n \geq 3$, there is at least one $\mathrm{SO}(n_i)$-Higgs bundle in the decomposition. If we consider this summand (E, Q) together with the one of the form $W_{-1} \to \mathcal{O} \to W_1$, we obtain a smooth $\mathrm{SO}_0(1, n_i+2)$-Higgs bundle. Using the same argument as in Theorem 8.4 we deduce that it is not a minimum (observe that $E \cong E^*$ and then it has weight zero). This implies that (V, Q_V, W, Q_W, η) is not a minimum and we conclude.

2. If $(\mathcal{O}, W, Q_W, \eta)$ is a strictly polystable $\mathrm{SO}_0(1,n)$-Higgs bundle (n odd) with $\eta \neq 0$ which is a fixed point of the circle action, it decomposes as a sum of a smooth minimum in $\mathcal{M}(\mathrm{SO}_0(1,2))$ of the form

$$W_{-1} \to \mathcal{O} \to W_1,$$

together with a sum of $\mathrm{SO}(n_i)$-Higgs bundles with and at least one summand of one of the following types: an $\mathrm{SO}(2)$-Higgs bundle or a $\mathrm{U}(n_i)$-Higgs bundle. The existence of this summand in the decomposition is necessary to guarantee the strict polystability of $(\mathcal{O}, W, Q_W, \eta)$.

Since n is odd, $n-2$ is also odd, and since

$$\mathrm{U}(n_i) \hookrightarrow \mathrm{SO}(2n_i) \hookrightarrow \mathrm{SO}_0(1, n-2),$$

with $2n_i$ even, there is at least one $\mathrm{SO}(n_i)$-Higgs bundle (E, Q) in the decomposition (and n_i is odd).

As in the stable but non-simple case, if we consider this summand (E, Q) together with the one of the form $W^i_{-1} \to \mathcal{O} \to W^i_1$, we obtain a smooth $\mathrm{SO}(1, n_i+2)$-Higgs bundle which is not a minimum and we conclude. $\square$

Remark 9.2. If n is even, we can not guarantee the existence of an $\mathrm{SO}(n_i)$-Higgs bundle in the decomposition in the second part of the proof and then this result can not be generalized to the even case.

Using the characterization of the minima given by Theorem 9.1 we solve the problem of counting the connected components of the moduli space $\mathcal{M}(\mathrm{SO}_0(1,n))$ with n odd.

Theorem 9.3. *The moduli space of* $\mathrm{SO}_0(1,n)$*-Higgs bundles when* $n>1$ *is odd has* 2 *connected components.*

Proof. The topological invariant associated to an $\mathrm{SO}_0(1,n)$-Higgs bundle $(\mathcal{O}, W, Q_W, \eta)$ with $n \geq 3$ is the Stiefel-Whitney class $w_2 \in \pi_1(\mathrm{SO}(n,\mathbb{C})) \cong \mathbb{Z}_2 = \{0,1\}$. From Theorem 9.1 we have that, when n is odd, there are no minima of the Hitchin function with non-zero Higgs field, and then $\mathcal{M}(\mathrm{SO}_0(1,n))$ (n odd) is the disjoint union of the moduli spaces $\mathcal{M}_0(\mathrm{SO}_0(1,n))$ and $\mathcal{M}_1(\mathrm{SO}_0(1,n))$, which are connected. □

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